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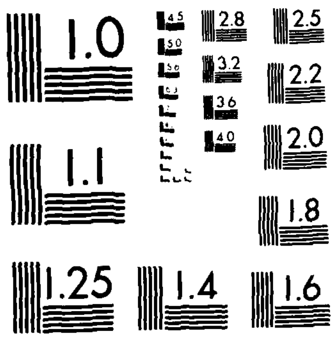

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V. F. Demyanov[1], C. Lemarechal[2] and J. Zowe[3]

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ABSTRACT

Given a multi-valued mapping F, we address the problem of finding another multi-valued mapping H that agrees locally with F in some sense. We show that, contrary to the scalar case, introducing a derivative of F is hardly convenient. For the case when F is convex-compact-valued, we give some possible approximations, and at the same time we show their limitations. The present paper is limited to informal demonstration of concepts and mechanisms. Formal statements and their proofs will be published elsewhere.

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Work Unit Number 5 (Optimization and Large Scale Systems)

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Part of this research was performed at the Mathematics Research Center of the University of Wisconsin-Madison, and sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

This paper is concerned with extension of the concept of derivative from functions to multi-valued mappings. Proper definitions of such extensions are useful to solve inclusions, and more specifically to minimize convex functions. Simple examples are given to show the difficulties, and some proposals are made to overcome them.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON APPROXIMATING A SET-VALUED FUNCTION LOCALLY

V. F. Demyanov[1], C. Lemarechal[2] and J. Zowe[3]

1. INTRODUCTION

Consider first the problem of solving a nonlinear system:

$$f(x) = 0 \tag{1}$$

where f is a vector-valued function. If we find a *first order approximation* of f near x, i. e. a vector-valued bi-function h such that

$$h(x;d) = f(x+d) + o(d) \tag{2}$$

(where $o(d)/\|d\| \to 0$ when $d \to 0$) then we can apply the *Newton principle:* given a current iterate x, solve for d

$$h(x;d) = 0 \tag{3}$$

(supposedly simpler than (1)) and move to $x+d$.

Everybody knows that if f is differentiable and if, in addition to satisfying (2), h is required to be affine in d, then it is *unambiguously* defined by

$$h(x;d) := f(x) + f'(x)d \tag{4}$$

Merging (2) and (4) and subtracting f(x) gives also a nonambiguous definition of f' (the jacobian operator of f) by:

$$f'(x)d := f(x+d) - f(x) + o(d).$$

Suppose now that we have to solve

$$0 \in F(x) \tag{5}$$

where F is a multi-valued mapping, i. e. $F(x) \subset R^n$. A possible application of (5) is in nonsmooth optimization, when F is the (approximate) subdifferential of an objective function to be minimized. To apply the same principle as in the single valued case, F(x+d) must be approximated by some *set*

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$H(x;d) \subset R^n$. Continuing the parallel and requiring H to be affine in d (whatever it means), we must express it as a *sum of two sets:* $H(x,d) = F(x) + G$. In summary, we want to find a set G such that, for all $\varepsilon > 0$ and $\|d\|$ small enough:

$$F(x+d) \subset F(x) + G + \varepsilon\|d\| U \tag{6.a}$$

and

$$F(x) + G \subset F(x+d) + \varepsilon\|d\| U \tag{6.b}$$

where U is the unit ball of R^n. Unfortunately, such a writing is already worthless. First, it does not help defining the "linearization" G: just because the set of subsets is not a group, F(x) cannot be substracted in (6). Furthermore, (6) is extremely restrictive: for n = 1, consider the innocent mapping $F(x) := [0,3x]$ (defined for $x \geqslant 0$). Take $x = 1$, $\varepsilon = 1$ and $d < 0$. It is impossible to find a set G satisfying (6.b), i. e. $[0,3] + G \subset [d,3+2d]$. For example, $G = \{d\}$ is already too "thick".

A conclusion of this section is that a first order approximation to a multivalued mapping cannot be readily constructed by a standard linearization; the definition of such an approximation is at present ambiguous. For a deep insight into differentiability of sets, we refer to [6] and its large bibliography. Here, for want of a complete theory, we will give in the next sections two possible proposals. None of them is fully satisfactory, but they are rather complementary, in the sense that each one has a chance to be convenient when the other is not. We will restrict ourselves to the convex compact case. Furthermore, as is usual in nondifferentiable optimization, we will consider only directional derivatives. Therefore we adopt simpler notations: x and the direction d being fixed, we call F(t) the image by F of $x + td$, $t \geqslant 0$. We say that H *approximates* F *to* 1st *order near* $t = 0^+$ if for every $\varepsilon > 0$, there is $\delta > 0$ such that $t \in [0,\delta]$ implies

$$F(t) \subset H(t) + \varepsilon t\, U \text{ and } H(t) \subset F(t) + \varepsilon t\, U \tag{7}$$

Note that, among others, F approximates itself!

2. MAPPINGS DEFINED BY A SET OF CONSTRAINTS

As a first illustration, suppose F is defined by:

$$F(t) := \{z \in R^n \mid c_j(t,z) \leqslant 0 \;\; \textit{for} \;\; j = 1,\dots,m\}$$

where the "constraints" c_j are convex in z. Assume the existence of $c'_j(0,z)$, the right derivative of $c_j(\cdot,z)$ at $t = 0$ ($c'_j(0^+,z)$ would be more suggestive). Then it is natural to consider approximating F(t) by

$$H(t) := \{z \mid c_j(0,z) + t\, c'_j(0,z) \leqslant 0 \;\; \textit{for} \;\; j = 1,\dots,m\}. \tag{8}$$

An algorithm based on this set would then be quite in the spirit of [7].

It is possible to prove that the H of (8) does satisfy (7), provided some hypotheses hold, for example

(i) $[c_j(t,z) - c_j(0,z)]/t \rightarrow c_j'(0,z)$ *uniformly in* z, *when* $t \downarrow 0$,

(ii) *there exists* z_0 *such that* $c_j(0,z_0) < 0$ *for* $j = 1,\ldots,m$.

A weak point of (8) is that it is highly non-canonical. For example, perturbing the constraints to $(1 + a_j t)c_j(t,z)$ gives the same F but does change H.

3. A DIRECT SET-THEORETIC CONSTRUCTION

If we examine (6) again, we see that there would be no difficulty if F(x) were a singleton: then (6) would always be consistent because F(x+d) would never be less thick than F(x), and F(x) could be subtracted. This leads to differentiating F at an *arbitrary but fixed* $y \in F(0)$. Define

$$F_y'(0) := \left\{ z \;\middle|\; \begin{array}{l} \textit{there exist } t_n \textit{ and } y_n \in F(t_n) \textit{ for } n \in \mathbb{N} \\ \textit{with } t_n \downarrow 0 \textit{ and } (y_n - y)/t_n \rightarrow z \end{array} \right\}$$

or, in a set-theoretic notation (see [2], Chapter VI):

$$F_y'(0) := \limsup_{t \downarrow 0} [F(t) - y]/t$$

This set is called the *contingent derivative* in [1], the (radial) *upper Dini derivative* in [6] and the *feasible set of first order* in [3]. We refer to [1] for an extensive study of F', but some remarks will be useful:

a) $F_y'(0)$ depends on the behaviour of F near y only. If we take an arbitrary $\alpha > 0$ and set $G(t) := F(t) \cap \{y + \alpha U\}$, then $G_y'(0) = F_y'(0)$.

b) If $F(t) \equiv F(0)$ does not depend on t, $F_y'(0)$ is just the tangent cone to F(0) at y.

c) Let A be a convex set in R^n, and $f : [0,1] \longrightarrow R^n$ a differentiable mapping (with $f(0) = 0$ for notational simplicity). Consider $F(t) := \{f(t)\} + A$. Given $y \in F(0) = A$, call T_y the tangent cone to $F(0) = A$ at y. Then it can be shown that $F_y'(0) = \{f'(0)\} + T_y$. This is the situation when F is the approximate subdifferential of a convex quadratic function (see [4]).

d) Let $n = 2$. Given $r \in R$, consider $F(t) := P(t) \cap U$ with the halfspace $P(t) := \{y = (y_1, y_2) \mid y_2 \geq r\,t\,y_1\}$. It can be shown that, for $y = 0 \in F(0)$, $F_0'(0) = \{z = (z_1, z_2) \mid z_2 \geq 0\}$; $F_0'(0)$ is the same as it would be if r were 0 (in which case F(t) would be fixed), and *does not predict the rotation* of F(t) around $y = 0$.

Because a convex set is the intersection of the cones tangent to it, our remark b) above suggests to approximate $F(t)$ by

$$H(t) := \cap \{y + t F'_o(0) \mid y \in F(0)\} \tag{9}$$

Of course, this will be possible only under additional assumptions (not only due to the multi-valuedness of F; for example $F(t) := \{t \sin \log t\}$ has $F(0) = \{0\}$, $F'_o(0) = [-1,+1]$ and $H(t) = [-t,+t]$).

Before mentioning the assumptions in question, we introduce another candidate to approximate F: for $p \in R^n$, denote by $s_p(t) := \sup \{<p,y> \mid y \in F(t)\}$ the support function of $F(t)$. It is known that F can be described in terms of s, namely $F(t) = \{y \mid <p,y> \leq s_p(t) \ \forall p \in R^n\}$. Then, if s_p has a (directional) derivative $s'_p(0)$, the following set is natural (see [5]):

$$G(t) := \{y \mid <p,y> \leq s_p(0) + t s'_p(0) \quad \forall p \in R^n\}. \tag{10}$$

To assess these candidates (9) and (10), the following assumptions can be considered:

(i) $[s_p(t) - s_p(0)]/t \to s'_p(0)$ uniformly for $p \in U$, when $t \downarrow 0$;

(ii) $F(0)$ has a nonempty interior.

They allow to prove:

If (i) holds, then $H(t) = G(t)$; if (ii) also holds, then (7) holds.

We remark that (i) alone suffices to prove the second half of (7), which is the important one for (5) (solving $0 \in H(t)$ gives some among the possible Newton iterates); however $H(t)$ may be void if (ii) does not hold. It is also interesting to remark that, if $s'_p(0)$ is assumed to be convex in p (in which case (ii) is not needed), then it is the support function of a convex set *that we are entitled to call* $F'(0)d$ because there holds $H(t) = F(0) + t F'(0)d$ (due to additivity of support functions). In other words, convexity of $s'_p(0)$ gives the "easy" situation in which (6) holds.

The role of assumption (i) is more profound. It is natural to require that $F'_y(0)$ does predict the behaviour of $F(t)$ near y; this behaviour is trivial when $y \in \text{int } F(0)$ (then $F(t)$ *must* contain y for all t small enough); if y is on the boundary of $F(0)$ then there is a normal cone $N_y(0)$ to $F(0)$ at y, and $s_p(0) = <p,y>$ for $p \in N_y(0)$; hence the behaviour of $F(t)$ near y is naturally related to the behaviour of $s_p(t)$ for these normal p's (incidentally, a key result is that $F'_y(0) = \{z \mid <p,z> \leq s'_p(0) \ \forall p \in N_y(0)\}$; (i) is essential for this). However, it is not only some technicalities in the proof that require the *uniformity* stated in (i), but rather the deficiency of F' suggested by d) above: consider the innocent mapping

$$F(t) := \{y = (y_1,y_2) \mid 0 \leq y_1 \leq 1, \ t y_1 \leq y_2 \leq 1\}.$$

Given $\alpha \in R$ and $p = (\alpha,-1)$, $s_p(t) = \max \{(\alpha-t)y_1 \mid 0 \leq y_1 \leq 1\}$ and thus, (i) is violated: when $\alpha \downarrow 0$, $s_p'(0)$ jumps from -1 to 0. For this example, $H(t) = G(t) = [0,1] \times [t,1]$, which is a poor approximation of $F(t)$. This is rather disappointing, but observe that Section 2 is well-suited for the present F.

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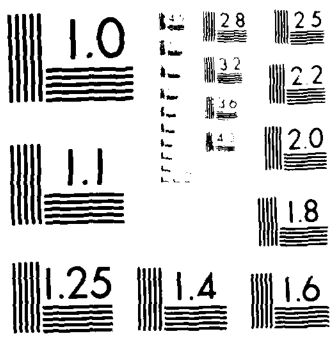

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